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Choosing the Right TOOL

Fourth graders use groupable base-ten manipulatives to learn division.

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The Common Core State Standards for Mathematics (CCSSM) drive the curricular and instructional decisions for most teachers in the United States (CCSSI 2010). Division in fourth grade is a major hurdle for many students, and CCSSM has placed the third-, fourth-, and fifth-grade division standards in the Number and Operations in Base-Ten (NBT) domain. *Base ten* is not a new term, but its emphasis in CCSSM has revived attention to this foundational characteristic of our place-value system. In addition to content standards, teachers of K–grade 12 must also implement eight Standards for Mathematical Practice (SMP). Two of these practices, *Use appropriate tools strategically* (SMP 5, CCSSI 2010, p. 7) and *Look for and make use of structure* (SMP 7, p. 8), guide teachers' choices of appropriate instructional strategies and materials.

FIGURE 1

Both (a) pregrouped and (b) groupable base-ten manipulatives can be made from easily attainable materials (e.g., colored copy paper and wooden craft sticks).

(a) Common *pregrouped* base-ten blocks are arranged in single squares (aka a *unit*), tens (aka a *rod*), and hundreds (aka a *flat*).

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(b) Ten craft sticks (aka a *bundle*), an example of *groupable* base-ten manipulatives, are secured by a rubber-band with ten bundles placed in a plastic baggie (aka a *baggie*).

FIGURE 2

Base-ten craft sticks are grouped using place-value concepts. A baggie represents the hundreds place and holds ten bundles. Each bundle has ten sticks, secured with a rubber band.

Students' success with fourth-grade content standards builds on mathematical knowledge learned in third grade and creates a conceptual foundation for division standards in subsequent grades that focus on the division algorithm. The division standards in fourth and fifth grade are similar; but in fourth grade, division problem divisors are only one digit; whereas in fifth grade, divisors are two digits. The division model explained in this article uses groupable base-ten manipulatives to teach the fourth-grade division standard (4.NBT.B.6). By carefully selecting the right tool for the instructional task, teachers can keep their students' focus on place value, allow for easy manipulation of the materials, and emphasize division's base-ten structure—all to build conceptual understanding.

Using manipulatives to teach division is not a new idea. Over fifty years ago, Dodds expressed concern with how division is taught:

> I am convinced that many difficulties that arise in . . . schools would never do so if from the very beginning division were taught differently. . . explaining and of putting on paper what we can "do in our heads" or by using bundles of sticks. (1960, p. 179)

Although "bundles of sticks" is mentioned, no elaboration is provided. Three recent articles in *TCM* (Cooper and Tomayko 2011; Sellers 2010; and Martin 2009) offered support for teaching division and place-value concepts. My article expands on these and explains how using *groupable* instead of *pregrouped* base-ten materials will better support students' conceptual and procedural understanding of division.

Two forms of base-ten manipulatives

The effectiveness of using manipulatives depends on a number of factors, one of which is selecting the most appropriate to support the mathematics to be learned (Brown, McNeil, and Glenberg 2009). In primary grades, set models like beans, counters, or cubes are often used to teach addition, subtraction, single-digit multiplication, and early division. However, these individual items do not have the base-ten structure necessary to teach more advanced levels of these concepts.

Manipulatives that have the base-ten structure come in two forms, *pregrouped* and *groupable* (see **fig. 1**) (see also Van de Walle, Karp, and Bay-Williams 2013, p. 196).

Groupable

Although both forms model our number system's 1:10 ratio, groupable materials—like bundles of sticks grouped in sets of 1, 10, and 100 (see **fig. 2**)—allow students to more easily perform the 1:10 "ungrouping" action referred to as *renaming* in this article. Exchanging one place value for another is as easy as emptying a baggie or removing a rubber band. Because renaming occurs frequently in division, being able to make exchanges easily and efficiently is important.

Pregrouped

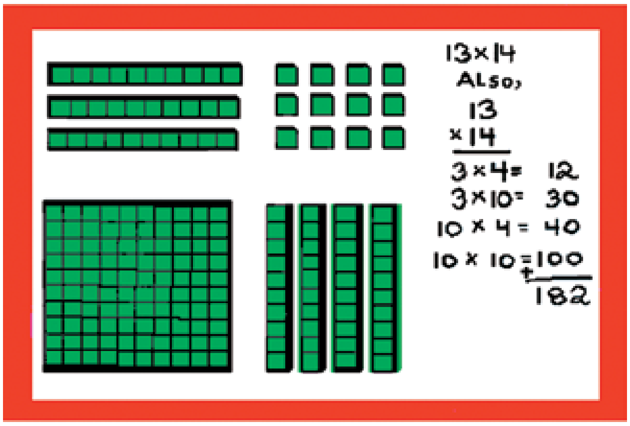
Pregrouped manipulatives like base-ten blocks have a static form that renders them less efficient when making place-value exchanges during division. However, they are very effective when teaching the area model of multiplication—the inverse operation of division—because they have the attribute of area (see **fig. 3**). When using the area model, exchanges between place values do not occur in the creation of the rectangle or when finding the areas of the four inner rectangles. Renaming occurs at the end of the process, when calculating partial products (see **fig. 3**). Therefore, using materials that can be easily *manipulated* is not an area model requirement.

Choosing the right tool when teaching division also allows teachers to show students sensitivity concerning the amount of *extraneous cognitive load* (Sweller, van Merrienboer, and Paas 1998) embedded in the task. This means that any extra cognitive demand placed on a student during an already challenging task could function as a barrier instead of a support to understanding. Using groupable base-ten manipulatives like craft sticks is one way to maintain *germane cognitive load* (Sweller, van Merrienboer, and Paas 1998), because the actions used with them are relevant to the mathematics that students are learning. Making sure that the materials align with the mathematical task offers a cognitive support instead of an impediment. For example, keeping the equivalence of two amounts clear during a

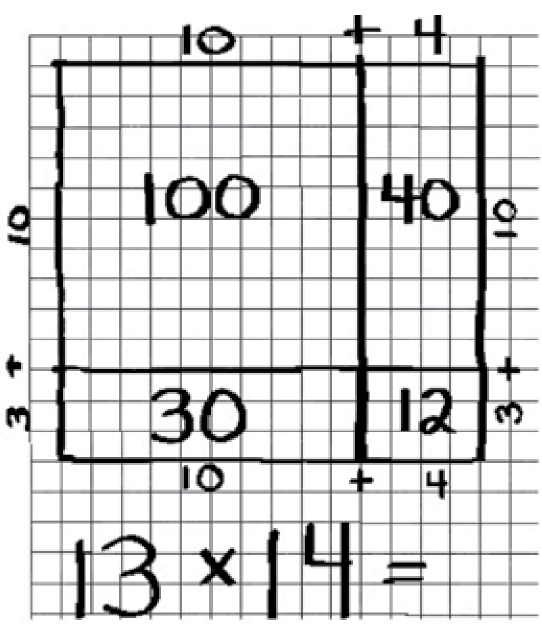
FIGURE 3

Base-ten blocks and centimeter paper are also easily attainable materials.

(a) Pregrouped base-ten blocks demonstrate the area for double-digit multiplication: Each factor of the problem is decomposed using sets of ten to align with the materials (13 = 10 + 3, or 1 rod and 3 units; 14 = 10 + 4, or 1 rod and 4 units). Partial products detail how area models center on place value and connect to the area model representation.



(b) Centimeter paper illustrates the physical materials and shows how the four smaller areas of the inner rectangles equal the area of the larger rectangle. An understanding of base-ten blocks allows for an easy observation of the product (i.e., 100 = 1 flat; 40 = 4 rods; 30 = 3 rods; and 12 =12 units). The only exchanges occur when 12 units are traded for 1 rod with 2 leftover units.



division exchange, such as one baggie for ten bundles (see **fig. 2**), is an example of germane cognitive load. By contrast, the same renaming action using pregrouped manipulatives would require students to count out ten units and swap them for one rod; and although one rod equals ten units, their physical equivalence is not as clear as the groupable form. Being forced to count out equivalent numbers of units adds extraneous cognitive load that may distract students. Students who are first learning division or who struggle to understand division would benefit from the more obvious representation that groupable materials provide.

Step 1: Making the groupable base-ten manipulative

Craft sticks, coffee stirrers, and straws are easily obtainable. The physical creation of the groupable manipulatives supports base-ten understanding. At the beginning of each school year, I give every student team a box of 1000 craft sticks and let them work together to assemble groups of 10 craft sticks, called a *bundle* (10), using a rubber band. Next, 10 bundles are put into a plastic baggie that seals, called a *baggie* (100) (see **fig. 2**). (Later in the fourth-grade year, when modeling a division problem with a four-digit dividend, sets of 10 baggies will be placed back into the original box, called a *box* [1000]. This article focuses on how to introduce the materials and solve a division problem with a single-digit divisor and a three-digit dividend.)

Step 2: Acting out *partition division* with groupable base-ten manipulatives

After teams create the manipulative sets, I give students five minutes to play with the materials. Doing so fulfills their natural curiosity and helps retain their attention to the activity. Then I intentionally select a problem with a divisor that will make use of the number of students within each team, because I want my students to act out the problem using the materials.

TABLE 1 These two models of division—partition and measurement—have equal groups.

Partition division	Measurement division
Also called *fair-sharing* or *partitive* division	Also called *repeated subtraction* or *quotitive* division
The number of groups is known, but the size of each group is unknown.	The size of each group is known, but the number of groups is unknown.
Example problems	
Carl brought twenty-four cookies to school and wants to share them fairly with his twelve friends (i.e., known number of groups). How many cookies (i.e., unknown size of the group) will each friend get? The number of items to be shared is twenty-four, and the number of groups (i.e., friends) is twelve. Only after the cookies are fairly shared in a "one for you, one for you, one for you" manner will it be apparent that each group received two cookies, the quotient of the problem. $24 \div 12 = 2$	Carl brought twenty-four cookies to school and wants to give a certain number of friends two cookies each (i.e., known size of each group). How many friends (i.e., unknown number of groups) will receive two cookies? The number of items to be shared remains the same (24), but instead of knowing how many people Carl is sharing the cookies among, you know the amount that each person will receive (2). Therefore, you are looking for the number of groups (i.e., friends) that sets of two cookies can be shared with before they run out. The quotient for this problem is twelve friends. $24 \div 2 = 12$

Students can physically use themselves as the "friends" mentioned in the problem or use sticky notes or index cards in the middle of the table to denote the "friends." Students, sticky notes, and index cards all model that the number of groups is known but the number within the groups is unknown.

After reviewing the two different models of division, partitioning and measurement (see **table 1**), we agree that a partition division problem exemplifies a "one for you, one for you, and one for you" action of passing out the sticks until they have all been distributed. Partition division is when you have a number of items to share, you know the number of groups the set is being shared among, but you do not know the number of items that each group will receive. When first teaching students how to divide craft sticks in their cooperative teams, the number of groups (i.e., friends) is known. This means that the problem scenarios used must be partition division because students represent the divisors. This differs from the measurement model, in which you know how many items each group will receive from the onset of the problem, but you do not know the number of groups that will receive the specified number of items. Instead of the "one for you, one for you, one for you" sharing action used in partition division, measurement division is a repeated subtraction action. I place the following problem on my document camera:

> Rae Anne has 238 sticks. She wants to share them equally among 3 friends (238 ÷ 3 = ?). How many will each friend get?

I ask teams to work together to determine a solution plan and record their actions. Each group has the materials they arranged in step 1: ten baggies, ten bundles in each baggie, ten sticks in each bundle. While walking around the room to listen to teams discuss their plans, I notice many teams pulling out three baggies and setting the rest aside. I ask one team to explain why it did this.

Avery: Rae Anne has 238 sticks to share. Two baggies is only 200 sticks and would not be enough. Three baggies is 300 sticks, which is plenty.

Teacher: Are you going to use all 300?

FIGURE 4

For the problem 238 ÷ 3, two baggies, three bundles, and eight sticks represent the base-ten place-value features.

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Avery: I am not sure [*pausing and looking at her friend Sam for help*] what to do with the extras.

Sam: We get to start the problem the way we want to. Let's just use two baggies, three bundles from the third baggie [*picking up the third baggie and removing three bundles*] and eight sticks from a different bundle in the third baggie [*removing the rubber band from a bundle in the third baggie and counting out eight sticks*]. This is what Rae Anne starts with: two whole baggies, three bundles, and eight sticks.

Teacher: Does everyone agree with Sam? [*Everyone nods.*] Now that you have what you are sharing, can you decide who you are going to share them with?

Kathy: The problem says that Rae Anne is sharing the sticks equally with three friends. It does not say that she is including herself in the sharing, so I think we need three groups [*putting three sticky notes in the middle of the table to represent three friends* (*see* **fig. 4**)].

Robert: This [*picking up two baggies*] isn't enough to share fairly with three friends.

Avery: This [*picking up the eight sticks*] is enough. Can we start with the sticks?

Kathy: I don't think we can start with the sticks, because they represent the ones place. Last year when we learned division, we always started with the largest place value. That would be the baggies in our problem.

Making sure that materials align with the mathematical task offers students cognitive support.

FIGURE 5

For the problem 238 ÷ 3, two baggies are renamed as twenty bundles and combined with the original three bundles to make twenty-three bundles to share.

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FIGURE 6

For the problem 238 ÷ 3, twenty-three bundles are shared among three friends, and each friend receives seven bundles with two leftover that will get renamed as twenty sticks in an upcoming step.

Avery: OK. Two baggies are not enough. What do you want to do?

Robert: [*Smiling and opening the two baggies*] Dump them!

Teacher: How many bundles are in front of you now? [*She points to the bundles that were dumped from the two baggies and the three bundles in the original problem.*]

Sam: We have twenty-three bundles. Now we have enough to share fairly with the three friends. [*See* **fig. 5**.]

Renaming is often poorly explained when teaching division using only procedural directions. The team determined what to do when not enough grouped craft sticks were available to be shared fairly. Dumping out the sticks physically represented what we do in a division algorithm when we look at the hundreds place (2), decide it is not large enough, and then look at both the hundreds and tens places and make a new number (23). What is often insufficiently explained when teaching the procedure in isolation is that it is not twenty-three *ones* but twenty-three *tens*. The groupable base-ten model clarifies that for students.

Sam: One for you, one for you, one for you [*passing out the twenty-three bundles to the three friends* (*i.e., the sticky notes*)], he has two bundles left. [*See* **fig. 6**.]

Robert: We need to rename the two bundles just like we renamed the two baggies. Two bundles is equal to twenty sticks. Remove the rubber bands, Sam.

Sam: [*Removing the rubber bands, he picks up the eight sticks from the original problem and combines them with the renamed twenty sticks from the two bundles. He shows the group the twenty-eight sticks before passing them out.*] One for you, one for you, one for you [*passing them out until he has one left in his hand and motions that he can go no further*]. Now what? [*See* **fig. 7**.]

Avery: It's a remainder.

Robert: Right; it's a leftover.

Kathy: It can be still be shared. Break it, Sam.

Sam: [*Getting the teacher's attention—she is working with a different group nearby—he*

holds up the one leftover stick to show the teacher, who approaches and looks at Kathy as Sam speaks.] She wants me to break your stick.

Teacher: [*Smiling*] Why do you want to break the leftover stick?

Kathy: This is our drawing [*pointing to what she has drawn to show how the team has kept track of renaming and sharing the sticks up to that point in the problem*]. [*See* **fig. 8**.] We can't share the one leftover stick fairly, though. If we can break the stick, we can give each friend one-third of the whole stick. Are we allowed to break it?

Sam: [*Shaking his head and disagreeing with Kathy's idea*] We always write a remainder next to the answer [i.e., the quotient] in a division problem [*writing* R. 1 *on a piece of paper*]. Isn't this stick just an R. 1?

Kathy: Yes [*pointing at the drawing again*], but if I break the stick into three equal pieces, I am renaming it like we did with the two baggies and the two bundles. One stick equals three one-third pieces. Each friend could get one of the pieces, because each piece is one-third of the stick. [*See* **fig. 9**.]

Sam: Oh, I see it. Each friend got seventy-nine whole sticks and one-third of another stick, not seventy-nine sticks and another whole stick. I was confused. Seventy-nine plus one more would be eighty, but I knew that was not right. The *R.* 1 does not mean you add another whole stick to what you already have. It means that the *R. 1* is split between the three friends equally. Each friend gets seventy-nine sticks and one-third of another.

Teacher: Does everyone agree? [*All members nod in agreement.*]

Step 3: Connecting the manipulatives to the algorithm

The teacher now addresses the entire class:

> Each team was successful in sharing the grouped sticks among three friends. Some teams drew pictures [*see* **fig. 8**], some teams wrote numbers alongside their pictures [*see* **fig. 10**], and some teams stood up and used the team members to represent the friends in the problem, physically acting out the problem. All teams reported that each friend

FIGURE 7

For the problem 238 ÷ 3, after the sticks have been shared fairly, each friend received seventy-nine sticks (seven bundles and nine sticks), but one stick still remains and has not yet been shared.

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FIGURE 8

Some students drew the problem 238 ÷ 3 using images of their groupable base-ten manipulatives: two baggies, three bundles, and eight sticks shared among three friends.

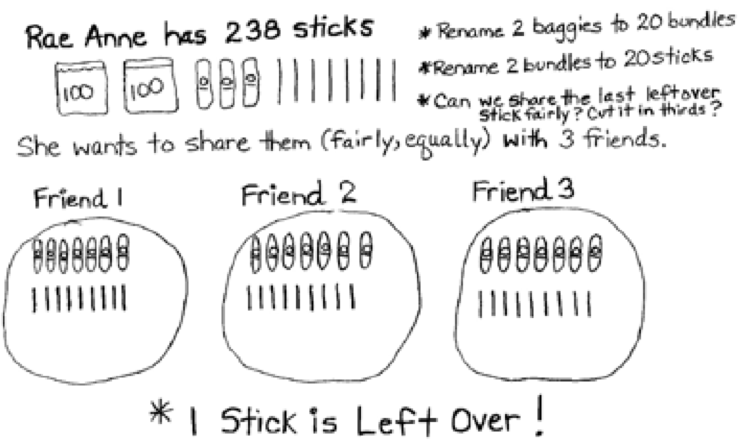
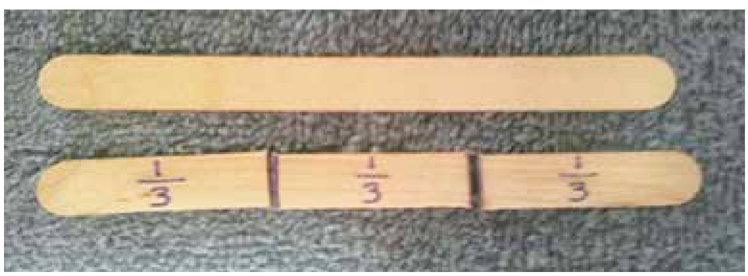


FIGURE 9

For the problem 238 ÷ 3, after the sticks have been shared fairly, one stick still remains. Instead of stopping the problem and recording "R1," the one stick can be cut into three equal fractional pieces, representing one-third of a whole stick.



Each step and decision made with the grouped sticks connects to choices that must be made using the division algorithm.

FIGURE 10

Some students drew the problem 238 ÷ 3 using images and numbers.

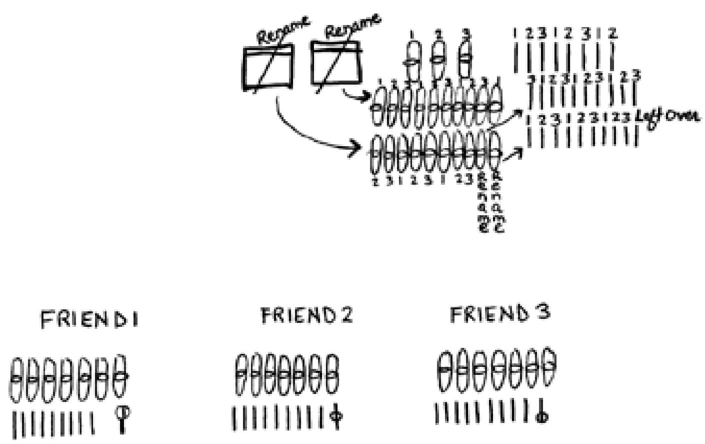
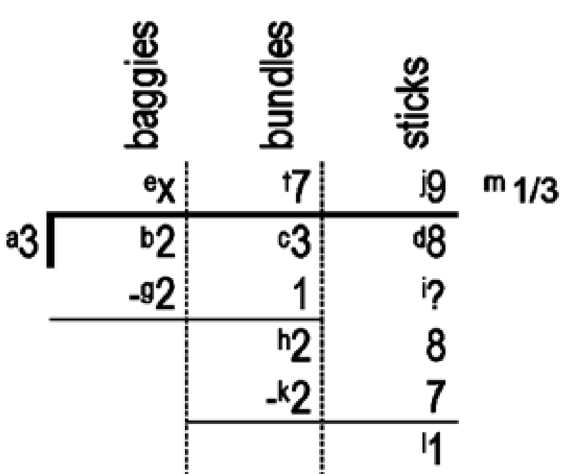


FIGURE 11

The actions with the grouped craft sticks correspond to the steps of the division algorithm.



received seventy-nine sticks. Four teams wrote the leftover stick as the fraction one-third. Two teams kept the remaining stick as a leftover and did not share it, keeping the notation of R. 1 that you all learned last year.

On the screen [*see* **fig. 11**] is a different way of writing the steps to solve the problem: 238 ÷ 3. Each tiny letter to the left of a number represents a step you just acted out with your craft sticks. For our next part, you can still use your baggies, bundles, and sticks. On your paper, write the letters *a* through *m* and explain what each letter represents in how you shared 238 sticks with 3 friends. For example, the first letter *a* is next to the number 3. In our problem, what did 3 represent?

Kathy: Our team used three sticky notes for the three friends in the problem.

Eric: I was the person who passed out the sticks in my team. My three other group members acted as the three friends.

Sam: My team also cut the leftover stick into three pieces at the end.

Teacher: You are all correct. Sam, you touched on a more advanced topic that we will be learning about soon. I want you to remember what you said, because it is very important. I am going to be using it in an upcoming lesson to help everyone understand how to interpret remainders. [*Sam smiles.*] If there are no additional comments or questions, you all may begin.

The teacher visited each group and recorded responses (see **table 2**) that teams wrote to share with the whole class when they came back together. Situating the activity in a cooperative group structure supplied a necessary scaffold for this task. The teacher continued to allow students to use grouped craft sticks

TABLE 2

A worked example compares the actions with the grouped craft sticks and the corresponding actions within the algorithm.

Algorithm identifier	Action taken with the grouped craft sticks
a	Number of people you are sharing the sticks with
b	"Two baggies" is equivalent to 200.
c	"Three bundles" is equivalent to 30.
d	"Eight sticks" is equivalent to 8.
e	(Always begin with your largest place value.) Question 1: "I have two baggies. Is this enough to share fairly with three people?" No. Place an "*x*" in the quotient above the b2 to prevent accidentally placing another number there later.
f	Comment and question 2: "Since I cannot share two baggies fairly, I need to rename them as twenty bundles. I already have three bundles, so I will combine the twenty and three bundles and share twenty-three bundles (notice the 23 in the dividend among three people). Is that enough?" Yes. Each of the three people get seven bundles.
g	"Twenty-one" is the total number of bundles fairly shared among three people.
h	"Two" is the number of leftover bundles. I cannot share them fairly with three people, so I need to rename them as twenty sticks.
i	Comment and question 3: "After renaming the leftover two bundles as twenty sticks and combining them with my original eight sticks, I have twenty-eight sticks to share fairly with three people. Do I have enough?" Yes.
j	"Nine" is the number of sticks each of the three people get.
k	"Twenty-seven" is the total number of sticks distributed.
l	"One" is the number of leftover sticks. It is not enough to share fairly with three people, so it becomes a remainder.
m	"One-third" is the fraction of the leftover stick each person gets.

to solve division problems and taught students how to use diagrams to represent their physical actions when they better understood the algorithm. She always made the materials available, though, and some students took materials home to use with homework assignments.

Choosing the right tool

Each step and decision made with the grouped sticks connects to choices that must be made using the division algorithm. After students have learned how to partition manipulatives by focusing on base-ten place values, a deeper understanding about grouping and renaming can occur. Learning this conceptual understanding in fourth grade will reinforce procedural knowledge to be learned in sixth grade.

This model can also be adapted in a number of ways to meet the needs of diverse learners. When differentiating instruction for those who require additional support, teachers can use a simpler division problem (e.g., a two-digit dividend and a one-digit divisor) having no remainder, to help students become more confident using the materials. For students who enjoy more challenging problems, changing

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the base from base-ten to another base, such as base six, could provide an academically meaningful experience to reinforce the concept of *base* in a number system while extending the learning opportunity. Finally, after students have become comfortable sharing grouped craft sticks representing the hundreds place (baggie), tens place (bundle), and ones place (stick), teachers can introduce an additional manipulative, the *box* of 1000 sticks, to expand place value to include the thousands place, which is also in the fourth-grade CCSSM (CCSSI 2010).

Using affordable materials like craft sticks to teach division content standards can help students build conceptual understanding. This groupable base-ten model also exemplifies two SMP, *Use appropriate tools strategically* (SMP 5, CCSSI 2010, p. 7) and *Look for and make use of structure* (SMP 7, p. 8). "Choosing the Right Tool" clarifies how the structure of our base-ten number system influences division and provides a foundation for future instruction on the division algorithm.

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Common Core Connections

4.NBT.1
4.NBT.6

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